An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations

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The nonlinear Schrödinger equation

We are interested in nonzero solutions of the PDE

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \qquad \text{(NLS)}$$

where $p > 2$ and λ are real parameters and $u : \Omega \to \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^N$ is an open domain.

For a given λ, solutions of (NLS) correspond to critical points of the *action functional* $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$J_\lambda(u) := \frac{1}{2}\int_\Omega \|\nabla u\|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x.$$

Action ground states

The functional J_λ is not bounded from below on $H_0^1(\Omega)$. Indeed, if $u \neq 0$, then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\Omega)} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\Omega)} - \frac{t^p}{p}\|u\|^p_{L^p(\Omega)} \xrightarrow[t\to\infty]{} -\infty.$$

A typical way to recover boundedness consists in introducing the *Nehari manifold* $\mathcal{N}_\lambda$:

$$\mathcal{N}_\lambda := \left\{ u \in H^1(\Omega) \setminus \{0\} \mid J_\lambda'(u)u = 0 \right\} = \left\{ u \in H^1(\Omega) \setminus \{0\} \mid \|\nabla u\|^2_{L^2(\Omega)} + \lambda \|u\|^2_{L^2(\Omega)} = \|u\|^p_{L^p(\Omega)} \right\}.$$

One then defines the *action ground state level*

$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

It is standard to show (see e.g. [8]) that, when Ω is bounded, $p < 2^*$ and $\lambda > -\lambda_1(\Omega)$, minimizers of the above problem exist and are *least action solutions of the problem*. They have a constant sign since one may replace u by $|u|$ in the minimization problem.

Nodal action ground states

All sign-changing solutions of the problem belong to the *nodal Nehari set*

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda \right\}$$

where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$ are the positive and negative parts of u.

Then, one considers the level

$$\mathcal{J}^{nod}(\lambda) := \inf_{u \in \mathcal{N}_\lambda^{nod}} J_\lambda(u).$$

When Ω is bounded, $p < 2^*$ and $\lambda > -\lambda_2(\Omega)$, minimizers of the above problem exist (see [1]) and are *least action nodal solutions of the problem.*

Normalized solutions

A *normalized solution* to (NLS) is a solution whose L^2-norm (usually called the *mass*) is prescribed a priori, whereas λ is an unknown of the problem.

They correspond to constrained critical points of the *energy* functional $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)}$$

on the L^2-sphere

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \mid \|u\|^2_{L^2(\Omega)} = \mu \right\},$$

λ arising then as a Lagrange multiplier.

Least energy normalized solutions, i.e. functions $u \in \mathcal{M}_\mu$ solving (NLS) and satisfying

$$E(u) = \inf\left\{ E(v) \mid v \in \mathcal{M}_\mu \text{ solves (NLS) for some } \lambda \in \mathbb{R} \right\}$$

are particularly interesting. When $p < 2 + \frac{4}{N}$, least energy solutions can be found (see e.g. [2]) by solving the minimization problem $\inf_{u \in \mathcal{M}_\mu} E(u)$.

When $p > 2 + \frac{4}{N}$, the energy E is unbounded from below on $\mathcal{M}_\mu$ for every μ, and different approaches are needed (e.g. of mountain-pass type, see [5]). This problem received a lot of interest on $\mathbb{R}^N$ but, so far, much less on bounded domains (see e.g. [6] and [7] however).

Two questions

How to find least energy normalized solutions in the L^2-supercritical regime?

How to find normalized nodal solutions?

Main results

Theorem. Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $p \in (2, 2^*)$, let

$$M_p(\Omega) := \left\{ \|u\|^2_{L^2(\Omega)} \mid u \in \mathcal{N}_\lambda \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\},$$

$$M_p^{nod}(\Omega) := \left\{ \|u\|^2_{L^2(\Omega)} \mid u \in \mathcal{N}_\lambda^{nod} \text{ and } J_\lambda(u) = \mathcal{J}^{nod}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

be the set of masses of all action ground states and nodal action ground states. Then

(i) if $p < 2 + \frac{4}{N}$, then
$$M_p(\Omega) = M_p^{nod}(\Omega) = (0, \infty);$$

(ii) if $p = 2 + \frac{4}{N}$, then there exist $0 < \mu_p, \mu_p^{nod} < \infty$ such that
$$(0, \mu_p) \subseteq M_p(\Omega) \subseteq (0, \mu_p] \quad \text{and} \quad (0, \mu_p^{nod}) \subseteq M_p^{nod}(\Omega) \subseteq (0, \mu_p^{nod}];$$

(iii) if $p > 2 + \frac{4}{N}$, then there exist $0 < \mu_p, \mu_p^{nod} < \infty$ such that
$$M_p(\Omega) = (0, \mu_p] \quad \text{and} \quad M_p^{nod}(\Omega) = (0, \mu_p^{nod}].$$

Theorem. Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and either

(i) $p < 2 + \frac{4}{N}$ and $\mu > 0$; or

(ii) $p = 2 + \frac{4}{N}$ and $\mu < 2\mu_N$, where $\mu_N := 2\inf_{u \in \mathcal{N}_1(\mathbb{R}^N)} J_1(u)$; or

(iii) $p > 2 + \frac{4}{N}$, Ω is star-shaped, and μ is small enough.

Then there exists a *least energy normalized nodal solution* with mass μ.

An idea of the techniques

We make a strong use of the *convex duality* between the action and the energy levels discovered in [4].

In particular, we show that if $\lambda_*, \lambda_*^{nod} \in \mathbb{R}$ are local minimizers of the maps

$$\lambda \mapsto \mathcal{J}(\lambda) - \frac{\mu}{2}\lambda \quad \text{or} \quad \lambda \mapsto \mathcal{J}^{nod}(\lambda) - \frac{\mu}{2}\lambda,$$

then action ground states in $\mathcal{N}_{\lambda_*}$ and nodal action ground states in $\mathcal{N}^{nod}_{\lambda_*^{nod}}$ have mass μ.

We then show that such minimizers exist under suitable assumptions on the masses.

A counterintuitive result

We take $p = 2 + \frac{4}{N}$ and we consider the ball. Noris, Tavares and Verzini have shown in [6] that the set

$$\left\{ \|u\|^2_{L^2(\Omega)} \mid u \text{ positive solution of (NLS) for some } \lambda \in \mathbb{R} \right\}$$

is equal to $(0, \mu_N)$. Our results imply that, for $\mu \in [\mu_N, 2\mu_N)$, there exist least energy normalized nodal solutions with mass μ. Thus...

Least energy solutions may be nodal!

References

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